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Parametrization of semi-dynamical quantum reflection algebra

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Abstract

We construct sets of structure matrices for the semi-dynamical reflection algebra, solving the Yang–Baxter-type consistency equations extended by the action of an automorphism of the auxiliary space. These solutions are parametrized by dynamical conjugation matrices, Drinfel'd twist representations and quantum non-dynamical R-matrices. They yield factorized forms for the monodromy matrices.

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1. Introduction

The semi-dynamical reflection algebra (SDRA) was first formulated on a specific example in [1]. The general formulation, together with a set of sufficient consistency conditions of Yang–Baxter type, was achieved in [2]. The transfer matrix, commuting trace formulae and representations of the comodule structures were defined in the same and in the following paper [3]; applications to the explicit construction of spin-chain-type integrable Hamiltonians were given in [4].

The generators of the SDRA are encapsulated in a matrix $K(\lambda)$ acting on a vector space $\mathcal{V}$ denoted by 'auxiliary space'. Two different types of auxiliary spaces will be considered here: either a finite-dimensional complex vector space V, or a loop space $V \otimes \mathbb{C}[[u]]$, with u the spectral parameter; in this last case the matrix $K(\lambda)$ should actually be denoted by $K(\lambda, u)$ and belongs to $\mathrm{End}(V) \otimes \mathbb{C}[[u]]$. This matrix $K(\lambda)$ satisfies the semi-dynamical reflection equation (SDRE):

$$A_{12}(\lambda)K_1(\lambda)B_{12}(\lambda)K_2(\lambda+\gamma h_1) = K_2(\lambda)C_{12}(\lambda)K_1(\lambda+\gamma h_2)D_{12}(\lambda) \qquad (1.1)$$

where A, B, C, D are c-number matrices in $\mathrm{End}(V) \otimes \mathrm{End}(V)(\otimes\mathbb{C}[[u_1, u_2]])$ depending on the dynamical variables $\lambda = \{\lambda_i\}_{i\in\{1\cdots N\}}$ and possibly on spectral parameters, this last dependence being then encoded in the labelling $(1, 2)$. When one considers (as in [5]) non-operatorial or so-called scalar solutions (i.e. dimension-1 representations of the algebra) this c-number solution matrix will be denoted by $k(\lambda)$. The exact meaning of the shift on these dynamical variables λ in (1.1) together with the main definitions and properties concerning the SDRA will be given in the next section and in appendix A.

The characteristic feature of the SDRA is that the integrable quantum Hamiltonians, obtained by the associated trace procedure from a monodromy matrix, exhibit an explicit dependence on the shift operators $\exp \partial_i$, $\left(\partial_i = \frac{\partial}{\partial \lambda_i}\right)$. In the case of the previously constructed dynamical reflection algebra known as 'dynamical boundary algebra' [6] however, such a dependence also arises but may altogether vanish when the basic scalar reflection matrix $k(\lambda)$ used to build the monodromy matrix is diagonal [4]. In the case of Gervais–Neveu–Felder dynamical quantum group, an explicit dependence also occurs but the commutation of Hamiltonians requires to restrict the Hilbert space of quantum states to zero-weight states under the characteristic Cartan algebra defining the dynamical dependence [7]. No such restriction occurs here which singles out the SDRA as the most useful algebraic framework to formulate spin-chain-type systems with an extra potential interaction between the sites of the spins and explicit dynamics on the positions, on the line of the spin-Ruijsenaar-Schneider systems [8]. Explicit formulae for these Hamiltonians, deduced in [4] in the most generic frame, yield complicated-looking objects with intricated connections between spin interactions and 'space-like' potential interactions. Such formulations may however simplify, as shall be shown here, when the building matrices A, B, C, D take some particular form.

Our purpose here is twofold. In order to construct consistent sets of A, B, C, D structure matrices we formulate generalized Yang–Baxter-type consistency equations (YBCE) extending the ones found in e.g. [9, 10] with the same assumption of associativity of the SDRA. This larger set of sufficient conditions is denoted by 'g-extended Yang–Baxter-type consistency equations' (g-YBCE) since they depend upon an automorphism g of the auxiliary space $\mathcal{V}$. Analysing and solving at least partially these two sets of equations (YBCE and g-YBCE) for the matrices A, B, C, D, we propose explicit parametrizations of the matrices A, B, C, D, the scalar solutions $k(\lambda)$ and the generating matrix $K(\lambda)$ in terms of quantum group-like algebraic structures (R matrices and Drinfel'd twists).

In a second step we plug these parametrizations into the general formulae for monodromy matrices, and obtain simplified expressions for them. These factorized expressions in terms of non-dynamical R matrices and Drinfel'd twists, simplify considerably the monodromy matrices found in [4] and represent therefore a suitable starting point to construct and solve quantum integrable Hamiltonians by allowing an explicit realization of the intricate formulae previously obtained in [4]. We also expect that this procedure may help to understand the nature of the algebraic structure implied by SDRE (1.1), specifically its possible connections with ordinary quantum group structures through Drinfel'd twists. However we must emphasize that at every stage, including the all-important first step of deriving Yang–Baxter-type consistency equations, but also A, B, C, D parametrizations, resolution of (1.1) for non-operatorial $k(\lambda)$ matrix, and even comodule structure yielding the monodromy matrix, we have proceeded by sufficient conditions; therefore we shall not cover here the full description of the algebraic content of (1.1).

Our paper goes as follows. In a first section we describe the notations, derive the sufficient Yang–Baxter-type consistency equations considered here and discuss the possible factorization of dynamical dependence in one of the four coefficient matrices. A second section treats the case of the simplest set ($g = \mathbb{1}$) of Yang–Baxter-type consistency equations, ending with the factorization of the monodromy matrix. We develop in an already extensive way the analysis of this set of YBCE, in order to establish clearly in a first stage the major steps of the parametrization procedure, and the subsequent derivation of the factorized form of the monodromy matrices, without the added complications induced by the existence of a non-trivial automorphism. We also discuss—more or less sketchily—some alternative paths to constructing different sets of solutions, by relaxing or eliminating some of the restrictions defining our sufficient conditions. A third section then deals with the full set of Yang–Baxter-type consistency equations for a generic g. The main features remain, but the occurrence of g induces several subtle effects, and requires the introduction of some supplementary assumptions, which we discuss in detail. Finally some conclusions and perspectives are drawn.

2. Notations and derivation of the two sets of Yang–Baxter-type consistency equations

The main features of the reflection equations yielding the SDRA are given in appendix A. In this section we will thus start with the SDRE (1.1), recall the definitions and properties of the objects it involves and obtain two sets of Yang–Baxter-type consistency equations (YBCE and g-YBCE).

We start by expliciting the exact meaning of the shift on the so-called dynamical variables λ in (1.1). Let $\mathfrak{g}$ be a simple complex Lie algebra and $\mathfrak{h}$ a commutative subalgebra of $\mathfrak{g}$ of dimension n. (For an extension to non-commutative $\mathfrak{h}$ see [11].)

Let us choose a basis $\{h^i\}_{i=1}^n$ of $\mathfrak{h}^*$ and let $\lambda = \sum_{i=1}^n \lambda_i h^i$, with $(\lambda_i)_{i\in\{1,\cdots,n\}} \in \mathbb{C}^n$ be an element of $\mathfrak{h}^*$. The dual basis is denoted in $\mathfrak{h}$ by $\{h_i\}_{i=1}^n$.[1] For any differentiable function $f(\lambda) = f(\{\lambda_i\})$ one defines

$$f(\lambda+\gamma h) = \mathrm{e}^{\gamma\mathcal{D}} f(\lambda)\, \mathrm{e}^{-\gamma\mathcal{D}}, \qquad \text{where} \quad \mathcal{D} = \sum_i h_i \partial_{\lambda_i}.$$

It can be seen that this definition yields formally

$$f(\lambda+\gamma h) = f(\{\lambda_i+\gamma h_i\}) = \sum_{m\geqslant 0} \frac{\gamma^m}{m!} \sum_{i_1,\ldots,i_m=1}^{n} \frac{\partial^m f(\lambda)}{\partial\lambda_{i_1}\ldots\partial\lambda_{i_m}} h_{i_1}\ldots h_{i_m}$$

which is a function on $\mathfrak{h}^*$ identified with $\mathbb{C}^n$ taking values in $\mathsf{U}(\mathfrak{h})$.

From now on, in order to alleviate the notations, we shall denote $f(h) \equiv f(\lambda+\gamma h)$.

Assumption of the associativity of SDRA and comparison of two possible ways of exchanging three K matrices requires zero weight conditions on structure matrices, namely:

$$\begin{aligned} &[h_i\otimes \mathbb{1}, B_{12}] = 0, \qquad [\mathbb{1}\otimes h_i, C_{12}] = 0, \\ &[h_i\otimes\mathbb{1} + \mathbb{1}\otimes h_i, D_{12}] = 0, \qquad \forall i\in\{1,\ldots,n\}. \end{aligned} \tag{2.1}$$

It then yields the Yang–Baxter-type consistency equations. A derivation of such sufficient consistency conditions yielding the YBCE is found (for the non-dynamical case) in e.g. [9] and for the semi-dynamical case in [2].

Here this derivation yields the following set of Yang–Baxter equations:

$$\begin{aligned} &\text{(a)} \quad A_{12}A_{13}A_{23} = A_{23}A_{13}A_{12} \\ &\text{(b)} \quad A_{12}C_{13}C_{23} = C_{23}C_{13}A_{12}(h_3) \\ &\text{(c)} \quad D_{12}B_{13}B_{23}(h_1) = B_{23}B_{13}(h_2)D_{12} \\ &\text{(d)} \quad D_{12}(h_3)D_{13}D_{23}(h_1) = D_{23}D_{13}(h_2)D_{12}. \end{aligned} \tag{2.2}$$

This set, obeyed for instance by the constant (i.e. non-spectral parameter dependent) A, B, C, D matrices [1] associated with the Ruijsenaar–Schneider (RS) A_n rational and

[1] This labelling of the dual basis must not be confused with the traditional labelling of auxiliary spaces in the global formulation of the SDRE (1.1).

trigonometric models [12], will be globally denoted as 'standard Yang–Baxter-type consistency equations' or YBCE. It is in fact the simplest example of a more generic form derived presently, but it is worth separating it in our derivation of parametrization of solutions, so as to treat it as a first simpler example even though it already exhibits the essential features of this parametrization.

A more general form of Yang–Baxter-type consistency equations is indeed derived from (2.2) once one notes that the identification of the structure matrices A, B, C, D in (1.1) exhibits some freedom due to the invariance of (1.1) under suitable transformations. In particular, the exchange algebra encapsulating the exchange relations for the generators of the SDRA building the matrix K (understood as an object in $\mathrm{End}\mathcal{V} \otimes \mathfrak{a}$ where $\mathfrak{a}$ is the SDRA) can be equivalently formulated by multiplying the lhs of (1.1) by $g \otimes \mathbb{1}$, where g is an automorphism of the auxiliary space $\mathcal{V}$ (see appendix A for notations on the auxiliary space).

Remark. The complete multiplication of (1.1) by *two* automorphisms $g \otimes g'$ can always be brought back to this form by a global change of basis on $\mathcal{V}$ parametrized by g', multiplying the rhs of (1.1) by $g'^{-1} \otimes g'^{-1}$, for g' any automorphism on $\mathcal{V}$. The endomorphisms h representing the generators of the Lie algebra $\mathfrak{h}$ acting on $\mathcal{V}$ (assumed to be a diagonalizable module of $\mathfrak{h}$) are accordingly redefined as $g' h g'^{-1}$.

In order to be able to undertake some specific technical manipulations, we shall restrict g, in the case when $\mathcal{V}$ is an evaluation module with spectral parameter u, by requiring that its adjoint action on any matrix in $(\mathrm{End} V^{\otimes N} \otimes \mathbb{C}[[u_1 \ldots u_N]])$ yields again a 'factorized' matrix in $(\mathrm{End} V^{\otimes N} \otimes \mathbb{C}[[u_1 \ldots u_N]])$. In other words the adjoint action of g must be compatible with the evaluation representation. This is equivalent to asking that, provided that g admits an operatorial logarithm $\gamma = \log g$, $[[\gamma, u.], u.] = 0$ where $u.$ is the automorphism of formal multiplication by u on $\mathcal{V}$. As an example, any automorphism γ commuting directly with u will provide a suitable $g = \exp \gamma$.

We shall also be later interested in particularizing endomorphisms γ such that $[\gamma, u.] = 0$. This is indeed equivalent to assuming that the action of γ on $\mathcal{V} = V \otimes \mathbb{C}[[u]]$ is represented by a functional matrix $M(\gamma) \in \mathrm{End} V \otimes \mathbb{C}[[u]]$ acting on $\mathcal{V}$. Such endomorphisms will be called 'factorizable' for obvious reasons. Automorphisms of the type $g = \exp\gamma$ with $[[\gamma, u.], u.] = 0$ will be called 'ad-factorizable'.

This lhs gauging of (1.1) now leads to a new definition of structure matrices:

$$\tilde{A}_{12} = g_1 A_{12} g_2^{-1}, \qquad \tilde{B}_{12} = g_2 B_{12}, \qquad \tilde{C}_{12} = g_1 C_{12}, \qquad \tilde{D}_{12} = D_{12}. \tag{2.3}$$

If we now assume consistently that $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$ (instead of A, B, C, D) obey the sufficient equations (2.2) we get a new set of Yang–Baxter-type consistency equations for A, B, C, D:

$$\begin{aligned}
&\text{(a)} \quad A_{12} A_{13}^{gg} A_{23} = A_{23}^{gg} A_{13} A_{12}^{gg} \\
&\text{(b)} \quad A_{12} C_{13}^{g_1} C_{23} = C_{23}^{g_2} C_{13} A_{12}^{gg}(h_3) \\
&\text{(c)} \quad D_{12} B_{13} B_{23}^{g_3}(h_1) = B_{23} B_{13}^{g_3}(h_2) D_{12} \\
&\text{(d)} \quad D_{12}(h_3) D_{13} D_{23}(h_1) = D_{23} D_{13}(h_2) D_{12}
\end{aligned} \tag{2.4}$$

where X_{12}^{g1}, X_{12}^{g2} and X_{12}^{gg} now denote respectively the following adjoint actions $g_1 X_{12} g_1^{-1}$, $g_2 X_{12} g_2^{-1}$ and $g_1 g_2 X_{12} g_1^{-1} g_2^{-1}$.

The generating matrix K is unmodified under this operation, and will thus be used directly when building monodromy matrices from the comodule structure. Consistency however will require to use tilded matrices (2.3) to build the N-site monodromy matrix. This set of equations is hereafter denoted by 'g-deformed Yang–Baxter-type consistency equations' or g-YBCE.

It is interesting to note that although the tilded 'structure matrices' are not obtained as adjoint actions of g on the c-number original matrices A, B, C, and may therefore not be represented as finite-size matrices in the evaluation representation when $\mathcal{V} = V \otimes \mathbb{C}[[u]]$, the new Yang–Baxter equations exhibit only adjoint actions of the automorphism g on the original c-number matrices A, B, C, hence are again written in terms of finite-size numerical matrices as follows from our restriction on g. On the example in [1] where $g = \exp[\frac{\mathrm{d}}{\mathrm{d}u}]$, u being the spectral parameter in the evaluation representation on $\mathcal{V} = V \otimes \mathbb{C}[[u]]$, it appears that in this case, although the structure matrices (2.3) are no longer c-number matrices (in other words, $V \otimes \mathbb{C}[[u]]$ is not an evaluation module for (2.3)) the Yang–Baxter equations themselves admit a representation (2.4) on the evaluation module, allowing the normal matrix manipulations to parametrize its solutions. Auxiliary action is here a shift of the spectral parameter.

We shall impose two further restrictions on g. The first is purely technical: we shall assume the existence of an endomorphism $\log g$ on $\mathcal{V}$ such that $\exp[\log g] = g$. This will be used later when solving the so-called quasi-non-dynamical conditions on given matrices acting on $\mathcal{V}$ or $\mathcal{V} \otimes \mathcal{V}$. The second one will impose that g does not depend on dynamical variables; it will play a central role when solving the Yang–Baxter equations.

It is finally relevant to start at once discussing the possible parametrizations of the D matrix which can essentially be treated (as will be seen in the next sections) independently of A, B, C. Analysing the possibilities of existence of invertible scalar (non-operatorial) solutions $k(\lambda)$ to (1.1) leads us to consider three possible situations for the relevant parametrizations of D. They will take a general form:

$$D_{12} = q_1^{-1} q_2^{-1}(\lambda + h_1)\tilde{R}_{12} q_{12}(\lambda + h_2) q_2 \tag{2.5}$$

where q is a scalar dynamical matrix in $\mathrm{End}V$ or $(\mathrm{End}V) \otimes \mathbb{C}[[u]]$ (factorizable). The three possibilities to consider are the following:

(1) Existence of decomposition (2.5) with a non-dynamical R-matrix $\tilde{R}$:

$$\tilde{R}_{12}(\lambda + h_3) = \tilde{R}_{12}(\lambda) \qquad \tilde{R}_{12}\tilde{R}_{13}\tilde{R}_{23} = \tilde{R}_{23}\tilde{R}_{13}\tilde{R}_{12}. \tag{2.6}$$

(2) Existence of a decomposition (2.5) with a quasi-non-dynamical R-matrix, i.e.

$$\begin{aligned} &\tilde{R}_{12}(\lambda + h_3) = f_1 f_2 \tilde{R}_{12}(\lambda)(f_1)^{-1}(f_2)^{-1} \\ &R^0_{12} R^{0 f_1 f_3}_{13} R^0_{23} = R^{0 f_2 f_3}_{23} R^0_{13} R^{0 f_1 f_2}_{12} \\ &\text{where} \quad R^0_{12} = Ad.\exp[-\sigma(\log f_1 + \log f_2)]\tilde{R}_{12}(\lambda) \\ &\text{so that} \quad R^0_{12}(\lambda + h_3) = R^0_{12} \text{ non-dynamical.} \end{aligned} \tag{2.7}$$

Here f is an ad-factorizable automorphism of $\mathcal{V}$, not necessarily identified with the automorphism g in (2.4).

(3) Neither decomposition exists.

Remark. Situations 1 and 2 may coexist, but we shall not establish if and when such a coexistence arises, it being not relevant for our specific purpose.

Possibility 1 (hereafter denoted by 'de-twisting of the D matrix') is indeed realized when the D-matrix is the representation of the universal R matrix for the quasi-Hopf algebra obtained by Drinfel'd twist of a Hopf algebra. $\tilde{R}$ is then the representation of the universal R-matrix for the Hopf algebra [13–15]. By extension of this notion we shall sometimes denote as 'twist' the shifted conjugation by q in (2.5) and 'twist matrix' the q matrix.

It has been recently proven [16] at the level of universal R matrices that D-matrices of weak Hecke type, associated with the A_n simple Lie algebra, could always be constructed as Drinfel'd twists of non-dynamical Cremmer–Gervais [17] R-matrices

$$D_{12} = g_1^{-1} g_2^{-1}(h_1) R^{CG}_{12} g_1(h_2) g_2. \tag{2.8}$$

However, even in the case of simple A_n Lie algebra (no spectral parameter) exhaustive resolutions of the dynamical Yang–Baxter equation show that non-weak-Hecke-type solutions exist [18]. In addition, the case of A_n^1 affine Lie algebra (naturally relevant when D depends on a spectral parameter) is not covered by the result in [16]. We shall hereafter be led to differentiate between the cases where D can be 'detwisted' as in (2.8), and cases where D cannot be written as in (2.8). This is in particular relevant to study the possible existence and precise constructions of invertible c-number solutions $k(\lambda)$.

Possibility 2 (hereafter denoted by 'quasi-detwisting' of D-matrix) has as far as we know no such interpretation yet, but should have a relation with the Drinfeld twist formulation in the context of the g-deformed Yang–Baxter equations.

We can now start the discussion on parametrization of A, B, C, D and K and construction of monodromy matrices and Hamiltonians, starting with the simpler case of standard Yang–Baxter-type equations (2.2).

3. Standard Yang–Baxter-type consistency equations

3.1. The A, B, C matrices

Once again $\mathcal{V}$ is either a finite-dimensional vector space V or an evaluation module $V\otimes\mathbb{C}[[u]]$. We assume that the vector space V is an irreducible representation of the dynamical algebra $\mathfrak{h}$. Since B_{12} is a space-1 zero weight matrix, and choosing from now on $\mathfrak{h}$ to be the Cartan algebra of $(gl(n))$, B can be parametrized as

$$B=\sum_{i=1}^{n} e_{ii}\otimes b_i(\lambda) \qquad b_i(\lambda)\in \mathrm{End}V\otimes\mathbb{C}[[u]], \tag{3.1}$$

Since D is a zero-weight matrix, it can be parametrized as

$$D=\sum_{i,j=1}^{n} d_{ij}(\lambda)e_{ii}\otimes e_{jj}+\sum_{i\neq j=1}^{n}\Delta_{ij}(\lambda)e_{ij}\otimes e_{ji}. \tag{3.2}$$

Equation (2.2c) now reduces to

$$d_{ij}(b_ib_j(h_i)-b_jb_i(h_i))=0. \tag{3.3}$$

We shall from now on, until the end of the paper, assume that all diagonal elements $d_{ij}\neq 0$, for all $i, j\in\{1,\dots,n\}$.

In this case $b_ib_j(h_i)=b_jb_i(h_i)$ for all i, j. If all b_i's are invertible $(n\times n)$ matrices, this implies that b_i are parametrized as:

$$b_i=b^{-1}b(\lambda_j+\delta_{ij}\gamma) \quad \text{with } b \text{ some invertible matrix} \tag{3.4}$$

or equivalently

$$B_{12}=\mathbb{1}\otimes b^{-1}b(h_1)=b_2^{-1}b_2(h_1) \tag{3.5}$$

using the compact dynamical shift notation and space indices. Here again $b(\lambda)\in \mathrm{End}V\otimes\mathbb{C}[[u]]$.

If some b_i's are not invertible the simple parametrization (3.4) is not available. Examples of such situations are easily given. Define a set of n mutually commuting projectors P_i, such that in addition $[P_i, b]=0$, then

$$b_i=P_ib^{-1}b(\lambda_j+\delta_{ij}\gamma) \tag{3.6}$$

obeys (3.3). It is not clear however whether an exhaustive classification of such solutions may be available.

If B is invertible, plugging back $C = B^{\pi}$ into (2.2b) yields the simple identity:

$$\left(b_1 b_2 A_{12} b_1^{-1} b_2^{-1}\right)(h_3) = b_1 b_2 A_{12} b_1^{-1} b_2^{-1} \tag{3.7}$$

equivalently stating that $b_1 b_2 A_{12} b_1^{-1} b_2^{-1} = R_{12}$ is non-dynamical. Furthermore plugging it into (2.2a) immediately implies that R_{12} is a non-dynamical solution of the Yang–Baxter equation, or a non-dynamical R matrix.

If B is non-invertible, the absence of explicit parametrization prevents us from deriving a general form for A. However the example (3.6) for instance is workable. Defining once again: $R_{12} = b_1 b_2 A_{12} b_1^{-1} b_2^{-1}$ yields from (2.2b)

$$R_{12} P_{i1} P_{i2} = P_{i1} P_{i2} R_{12}(h_i) \tag{3.8}$$

and from (2.2a) again the Yang–Baxter equation for R. Once again it may not be possible to exhaust all simultaneous solutions to Yang–Baxter equations and (3.8). However one deduces that if R is a non-dynamical R matrix and $\{P_i\}$ a set of projectors such that $[P_i \otimes P_i, R] = 0$ and $[P_i, b] = 0$ then they provide a consistent set of matrices

$$\begin{aligned} A_{12} &= b_1^{-1} b_2^{-1} R_{12} b_1 b_2 \\ B = C^{\pi} &= \sum e_{ii} \otimes P_i b^{-1} b(\lambda_i + \gamma). \end{aligned} \tag{3.9}$$

Such projectors exist e.g. if R is a Yangian-type solution in $A_n^1 \otimes A_n^1$

$$R = \mathbb{1} \otimes \mathbb{1} + \frac{\Pi_{12}}{\lambda - \mu} \tag{3.10}$$

since then for *any* projector $[P \otimes P, R] = 0$. Choosing these projectors P to commute with an arbitrary chosen matrix b, and with each other (e.g. elements among the set of projectors on eigenvectors of b) one gets A, B, and C.

To conclude: if $d_{ij} \neq 0$ for all i, j, and B invertible there exists a parametrization of A, B, C as:

$$\begin{aligned} A &= b_1^{-1} b_2^{-1} R b_1 b_2 \\ B = C^{\pi} &= \mathbb{1} \otimes b^{-1} b(\lambda + h_1) = b_2^{-1} b_2(\lambda + h_1) \end{aligned} \tag{3.11}$$

where R is a non-dynamical quantum R-matrix and b some dynamical matrix.

One immediately establishes here:

Proposition 2. *If A, B, C are parametrized as in (3.11) by matrices b and R, the following statements are equivalent*

- *(a) The SDYBE equation (1.1) has an invertible scalar solution $k(\lambda)$.*
- *(b) D can be de-twisted, following (2.8), to a non-dynamical matrix R with twist given by $q = bk$.*

Proof.

- (a) $\Rightarrow$ (b) by direct inversion of (1.1) yielding (2.8) with $q = bk$.
- (b) $\Rightarrow$ (a) by direct plug of (2.8) into (1.1) using q and b yielding $k = b^{-1} q$ as a scalar solution.

Hence, whether D cannot be detwisted at all or cannot be detwisted to R the absence of a scalar invertible solution may cause serious practical issues to build integrable spin-chain-type Hamiltonians. However, if D is de-twistable to another $\tilde{R}$, one may nevertheless draw interesting conclusions regarding possible non-invertible scalar solutions, and even monodromy matrices. We shall henceforth proceed with our general trichotomy.

3.2. The D matrix and K solutions

As indicated above, we shall separate this discussion into three subcases, whether or not D can be detwisted as in (2.8) and whether it is detwisted as in (2.6) or (2.7). Note immediately that one can show easily:

3.2.1. Cases 1 and 2. D is detwistable or quasi-detwistable. We use here the general form

$$D_{12} = q_1^{-1}(\lambda + h_2) q_2^{-1} \tilde{R}_{12} q_1 q_2 (\lambda + h_1) \tag{3.12}$$

where $\tilde{R}$ is either non-dynamical or quasi non-dynamical. If A, B, C are parametrized as in (3.11), plugging (3.11) and (3.12) into (1.1) leads to the following equation:

$$R_{12}(bKq^{-1})_1 q_1 (bKq^{-1})_2(h_1) q_1^{-1} = (bKq^{-1})_2 q_2 (bKq^{-1})_1(h_2) q_2^{-1} \tilde{R}_{12}. \tag{3.13}$$

General solutions to (3.13) are not obvious to formulate due to the coupling between spaces 1 and 2 induced by the adjoint action of $q_{1,2}$ on $(bkg^{-1})_{2,1}(h_{1,2})$. If however bKq^{-1} is such that:

$$(bKq^{-1})_1(h_2) = \mathcal{A} \otimes \mathbb{1} = \mathcal{A}_1 \tag{3.14}$$

for some matrix functional $\mathcal{A}$ then (3.13) simplifies to a Yang–Baxter-type form

$$R_{12}(bKq^{-1})_1 \mathcal{A} (bKq^{-1})_2 = (bKq^{-1})_2 \mathcal{A} (bKq^{-1})_1 \tilde{R}_{12}. \tag{3.15}$$

Condition (3.14) can be explicitly solved as follows. From the general definition of shifts, applied to the $gl(n)$ case, one has

$$(bKq^{-1})_1(h_2) = \sum_{i=1}^{n} bKq^{-1}(\lambda_j + \gamma\delta_{ij}) \otimes e_{ii}. \tag{3.16}$$

Factorizing $\mathbb{1}_2$ as in (3.14) requires to have

$$bKq^{-1}(\lambda_j + \gamma\delta_{ij}) = bKq^{-1}(\lambda_j + \gamma\delta_{lj}) \tag{3.17}$$

for any index pair (i, l). This is equivalent to restricting bKq^{-1} to depend on the following new set of dynamical variables

$$\sigma = \sum_{i=1}^{n} \lambda_i, \qquad \theta_i = \sigma - 2\lambda_i, \quad i = 2, \ldots, n \tag{3.18}$$

constrained by $bKq^{-1}(\theta_i + 2\gamma) = bKq^{-1}(\theta_i)$ for $i = 2, \ldots, n$.

Equation (3.13) now becomes a usual dynamical Yang–Baxter intertwining equation for $\kappa \equiv bKq^{-1}$ for the simplified situation where R itself is non-dynamical,

$$R_{12}\kappa_1(\sigma)\kappa_2(\sigma + \gamma) = \kappa_2(\sigma)\kappa_1(\sigma + \gamma)\tilde{R}_{12}. \tag{3.19}$$

We shall not discuss (3.19) in full generality. We now separate our discussion into two subcases.

3.2.2. D detwistable, $\tilde{R}$ non-dynamical. Two simple and relevant examples will now provide us with explicit realizations of solutions to the SDRE (3.13).

(1) Non-dynamical quantum group
Given any non-dynamical solution $\mathcal{Q}$ to:

$$R_{12}\mathcal{Q}_1\mathcal{Q}_2 = \mathcal{Q}_2\mathcal{Q}_1\tilde{R}_{12} \tag{3.20}$$

$K(\lambda) = b^{-1}\mathcal{Q}q(\lambda)$ realizes a solution of (3.13)). In particular if $\mathcal{Q}$ is a factorized matrix, represented in $\mathrm{End}V \otimes \mathbb{C}[[u]]$, $K(\lambda)$ is also such a solution to (1.1). It follows that:

a1. if $R = \tilde{R}$ ($\leftrightarrow$ existence of scalar invertible solution)

Any realization $\mathcal{Q}$ of the quantum group described by the RTT formulations with R as evaluated R matrix, will provide a realization of the SDRA as $K = b^{-1}\mathcal{Q}q$. This includes as well scalar solutions (yielding scalar k matrices) or operator like solutions (representations of the quantum group by operators on some Hilbert space $\mathcal{H}$). In particular, $\mathcal{Q} = \mathbb{1}$ yields an invertible scalar solution $k = b^{-1}q$, consistent with proposition 2.

a2. if $R \neq \tilde{R}$ (no invertible scalar solutions)
Then any intertwiner matrix (scalar or operational) $\mathcal{Q}$:

$$R_{12}\mathcal{Q}_1\mathcal{Q}_2 = \mathcal{Q}_2\mathcal{Q}_1\tilde{R}_{12} \tag{3.21}$$

provides us with realizations of the SDRA.

(2) Quasi-non-dynamical quantum group
Let us consider the more general quadratic exchange relation:

$$R_{12}\mathcal{Q}_1(a\mathcal{Q}a^{-1})_2 = \mathcal{Q}_2(a\mathcal{Q}a^{-1})_1\tilde{R}_{12} \tag{3.22}$$

for some ad-factorizable automorphism a of the auxiliary space $\mathcal{V}$, such that $[a \otimes a, R] = [a \otimes a, \tilde{R}] = 0$. From any non-dynamical representation $\mathcal{Q}$ of this exchange algebra (scalar or operatorial) one can build a representation (scalar or operatorial) of the SDRA as:

$$K = b^{-1}(\lambda)(\exp[\sigma \log a]\mathcal{Q}\exp[-\sigma \log a])q(\lambda), \tag{3.23}$$

assuming the existence of a logarithm of a. This adjoint action transforms the dynamical shift on any dynamical parameter λ into a conjugation by a, yielding what we will call quasi-non-dynamical condition for $\tilde{q} = (\exp[\sigma \log a]\mathcal{Q}\exp[-\sigma \log a])$:

$$\tilde{q}(\lambda + h_2) = a\tilde{q}(\lambda)a^{-1} \otimes \mathbb{1}. \tag{3.24}$$

Once again, ad-factorizability of a ensures that (3.22) and (3.23) are finite-matrix algebraic equations on the auxiliary space V.

3.2.3. D quasi-detwistable, $\tilde{R}$ quasi-non-dynamical. Here one assumes that $\tilde{R}$ obeys (2.7) for some ad-factorizable automorphism f of $\mathcal{V}$. It is still possible to obtain explicit representations of (1.1) as modified versions of the representations given in the previous subsection. Namely the non-dynamical quantum group (NDQG) construction (a) is modified as follows: (3.20) becomes

$$R_{12}\mathcal{Q}_1\mathcal{Q}_2 f_2 = \mathcal{Q}_2\mathcal{Q}_1 f_1\tilde{R}^0_{12} \tag{3.25}$$

where $\tilde{R}^0$ is the non-dynamical part of $\tilde{R}$ extracted from (2.5)

$$\tilde{R}_{12}(\lambda) = Ad.\exp[-\sigma(\log f_1 + \log f_2)]\tilde{R}^0_{12} \tag{3.26}$$

and $K(\lambda)$ becomes

$$K(\lambda) = b^{-1}(\lambda)\mathcal{Q}\exp[-\sigma \log f]q(\lambda). \tag{3.27}$$

The Quasi-NDQG (b) is modified as follows: (3.22) becomes

$$R_{12}\mathcal{Q}_1(a\mathcal{Q}a^{-1})f_2 = \mathcal{Q}_2(a\mathcal{Q}a^{-1})f_1\tilde{R}^0_{12} \tag{3.28}$$

with the K matrix now being

$$K = b^{-1}(\lambda)(Ad.\exp[\sigma \log a]\mathcal{Q})\exp[\sigma \log f]q(\lambda). \tag{3.29}$$

Note that here no relation between the two automorphisms a and f needs to be assumed. However if f, although ad-factorizable, is not factorizable (see, e.g., $\log f \equiv \mathrm{d}/\mathrm{d}u$), equations (3.25) and (3.28) cannot be written as algebraic equations for finite-size matrices in $\mathrm{End}V \otimes \mathbb{C}[[u]]$, and the objects $\mathcal{Q}$, solutions of (3.25) and (3.28), may not be expandable in formal power series of the variable u; subsequent interpretation of K as a generating functional for some quantum algebra is then unavailable, and the correct interpretation of (1.1) in this context remains to be explicited.

3.2.4. Case 3. D non-de-twistable. One is here able to build new sets of realizations $K(\lambda)$ of the SDRA if one knows at least one (non-invertible!) solution $K(\lambda)$, from the left comodule structure, described as follows:

Proposition 3. *If $K_0(\lambda)$ is a solution of (1.1), and A, B, C are parametrized by R and b as in (3.11), from any solution of:*

$$R_{12}q_1q_2(h_1) = q_2q_1(h_2)R_{12} \tag{3.30}$$

such that, once again, $q_n(h_m) = \mathcal{A}(q)_n \otimes \mathbb{1}_m$ (indices n and m refer here to the labelling of auxiliary spaces in a multiple tensor product), one can build a solution $b^{-1}qbK_0$ of (1.1).

One recovers once again the equations in (3.20) or (3.22) (for $R = \tilde{R}$). Any solution to (1.1) can be dressed to another solution, using any representation of the quantum group, or even quasi non-dynamical quantum group.

However when D cannot be detwisted, one cannot simplify the formulation of the monodromy matrix derived even from the simplest comodule structure of SDRA; hence we shall not consider this case in the next section.

3.3. Monodromy matrices

We shall restrict ourselves to the case where A, B, C are parametrized by matrices R and g (no $d_{ij} = 0$), and D is detwistable to a non-dynamical R matrix. In addition we shall only construct the monodromy matrix corresponding to the simplest comodule realizations of the SDRA, i.e. realizations by A, B, C, D matrices themselves (the specific construction of new comodule realizations using the parametrizations derived here goes beyond the intended scope of this study). Moreover we shall also consider the simplest, i.e. non-dynamical, realizations of scalar k matrices (3.20), (3.21). Construction of monodromy matrices to yield commuting spin-chain-type Hamiltonians is mostly relevant from a physical point of view when the scalar solutions are themselves invertible. We shall nevertheless also consider the non-invertible, detwistable case as well, but once again only where D is detwisted to a non-dynamical R matrix.

3.3.1. Existence of invertible solutions k. We shall recall that one can then parametrize A, B, C, D as

$$\begin{aligned} &\text{(a)}\quad B = C^{\pi} = \mathbb{1} \otimes b^{-1}b(\lambda + h_1) \\ &\text{(b)}\quad A = b_1^{-1}b_2^{-1}Rb_1b_2 \\ &\text{(c)}\quad D = (b_1k_1)^{-1}(h_2)(b_2k_2)^{-1}Rb_1k_1b_2k_2 \end{aligned} \tag{3.31}$$

where k is a particular invertible solution of (1.1). Other invertible solutions are given by:

$$\tilde{k} = b^{-1}\mathcal{Q}bk, \qquad \text{where } \mathcal{Q} \text{ is a scalar solution to} \qquad R_{12}\mathcal{Q}_1\mathcal{Q}_2 = \mathcal{Q}_2\mathcal{Q}_1R_{12}. \tag{3.32}$$

There may be other invertible solutions obtained by a general resolution of (3.13), but at this stage we have no explicit parametrization for them and we shall therefore restrict ourselves to the previous dressed solutions $b^{-1}\mathcal{Q}bk$.

We are now in a position to reformulate the monodromy matrix for a spin-chain-type model, obtained from the particular comodule structure of the SDRA and the quantum trace structures detailed in [2, 3], by plugging (3.31), (3.32) into the general formula. Denoting in addition by χ_0 the solution to the dual SDRE required to build a 'reflection' monodromy matrix, we recall that the N-site monodromy matrix can be chosen of either two forms, to yield local Hamiltonians [4] by a (partial) trace procedure over the finite vector space V whichever structure is chosen for the auxiliary space $\mathcal{V}$:

$$\chi_0^t A_{02N}C_{02N-1}\cdots A_{02}\big(h_<^{\text{odd}}\big)C_{01}T_0\big(h_<^{\text{odd}}\big)D_{01}B_{02}\cdots D_{02N-1}B_{02N}\,\mathrm{e}^{\mathcal{D}_0} \tag{3.33}$$

or $(A \to C, B \to D)$ making use of the first known comodule structure.

Remark: the notation $X_{0a}\big(h_<^{\text{odd}}\big)$ was introduced in [2] and denotes $X_{0a}\big(\lambda + \Sigma_{n=0}^{E(a/2)-1}h_{2n+1}\big)$.

One may also use as 'site' matrices $A \to (A^{-1})^T$, $B \to (B^{-1})^T$, $C \to (C^{-1})^T$, $D \to (D^{-1})^T$ but we shall not consider this alternative possibility here for the sake of simplicity. Note also the crucial occurrence of the shift operator $\exp[\mathcal{D}_0]$ in the formulation of the monodromy 'matrix'. This guarantees that partial traces of monodromy matrices over the finite vector space V commute as operators acting on the tensor product of the spin-chain Hilbert space (in this case $(\mathbb{C}^n)^{\otimes N}$) and the functional space of differential functions over $\mathfrak{h}^*$. The price to pay is that these traces lie not in the quantum reflection algebra defined by (1.1), but in the extended operator space containing in addition derivatives w.r.t. variables in $(\mathfrak{h}^*)^*$, such as built e.g. in [19]. It may be conjectured that the relevant traces operate not in a quantum group but in a quantum groupoid structure relevant to the dynamical Yang–Baxter algebras [11].

The monodromy matrix (3.33) then becomes

$$\mathcal{O}_N^{-1}(\sigma)\big\{\chi_0^t b_0^{-1}R_{02N}\cdots R_{02}\mathcal{Q}_0R_{01}\cdots R_{02N-1}b_0k_0\,\mathrm{e}^{\partial_0}\big\}\mathcal{O}_N(\sigma) \tag{3.34}$$

where the operator $\mathcal{O}_n(\sigma)$ acts only on the quantum spaces:

$$\mathcal{O}_N(\sigma) = b_{2N}b_{2N-1}k_{2N-1}(b_{2N-2})(h_{2N-1})\cdots b_1k_1(h_3+\cdots h_{2N-1}). \tag{3.35}$$

3.3.2. No invertible solutions, D detwistable to non-dynamical $\tilde{R}$. This corresponds to a situation where equation (3.31c) is replaced by

$$D_{12} = q_1^{-1}(h_2)q_2^{-1}\bar{R}_{12}q_1q_2(h_1) \tag{3.36}$$

but now $\bar{R}$ is a non-dynamical R-matrix not similar to R. In this case there exists no invertible scalar solution, otherwise D could be detwisted to R. This situation is not so interesting from the point of view of realistic physical model building of spin chains, but it yields once again an interesting reduction of the monodromy matrix and may help in disentangling the general structure of the semi-dynamical equation. Choosing the parametrization (3.31a), (3.31b), (3.36) and the scalar reflection solutions χ_0 and $\tilde{\chi}_0$ one gets a monodromy matrix:

$$\mathcal{O}_N^{-1}(\sigma)\left\{\tilde{\chi}_0^t b_0^{-1}R_{02N}\cdots R_{02}\left(\prod_k^{1\to N} q_{2k+1}\big(h_>^{\text{odd}}\big)b_0\chi_0\big(h_>^{\text{odd}}\big)q_0^{-1}\left(\prod_k^{1\to N} q_{2k+1}\big(h_>^{\text{odd}}\big)\right)^{-1}\right)\right.$$
$$\left.\times\,\bar{R}_{01}\cdots\bar{R}_{02N-1}b_0k_0\,\mathrm{e}^{\gamma\partial_0}\right\}\mathcal{O}_N(\sigma) \tag{3.37}$$

where

$$\mathcal{O}_N(\sigma) = \prod_{k}^{N \to 1} b_{2k}\big(h_>^{\mathrm{odd}}\big) q_{2k-1}\big(h_>^{\mathrm{odd}}\big). \tag{3.38}$$

If $b_0 \chi_0 q_0^{-1}$ is non-dynamical (i.e. if one chooses a solution χ_0 of type given in subset 3.2.2a), a factorized compact formula for the monodromy matrix is then yielded with a form analogous to (3.34). However one must be careful that since no invertible scalar solution χ_0 to (1.1) exists, one has *a priori* no relation expressing a given dual solution $\tilde{\chi}_0$ in terms of some direct solution χ.

This concludes our analysis of the semi-dynamical Yang–Baxter equation with ordinary Yang–Baxter conditions on A, B, C, D.

4. g-deformed Yang–Baxter-type consistency equations

We shall for this discussion restrict ourselves to the simpler situation where all diagonal terms d_{ij} of D are non-zero (as in section 3), but also where matrices B and C are immediately assumed to be invertible. Once again, in the case where $\mathcal{V}$ is chosen to be an evaluation module $(\mathrm{End}\, V \otimes \mathbb{C}[[u]])$ we assume that the adjoint action of the characteristic automorphism g on any operator represented by a finite-size matrix in $(\mathrm{End}\, V^{\otimes N} \otimes \mathbb{C}[[u_1 \ldots u_N]])$ gives again a finite-size matrix (ad-factorizability condition).

4.1. Parametrization of A, B, C

Consider (2.4c) with the conditions $d_{ij} \neq 0$, B invertible. Equation (3.3) is turned into

$$b_i g b_j(\lambda_i + \gamma) g^{-1} = b_j g b_i(\lambda_j + \gamma) g^{-1}. \tag{4.1}$$

Assuming all b_i's to be invertible, (4.1) is solved by

$$\begin{aligned} B_{12} &= b_2^{-1} g_2 b_2(\lambda + h_1) g_2^{-1} \\ C_{12} &= b_1^{-1} g_1 b_1(\lambda + h_2) g_1^{-1}. \end{aligned} \tag{4.2}$$

Defining now

$$R_{12} = b_1 g_2 b_2 g_2^{-1} A_{12} \big(g_1 b_1 g_1^{-1}\big)^{-1} b_2^{-1} \tag{4.3}$$

equation (2.4b) yields

$$R_{12}(\lambda) = g_1 g_2 R_{12}(\lambda + h_3) g_1^{-1} g_2^{-1} \tag{4.4}$$

meaning that for any index i

$$R_{12}(\lambda_i + 1) = g_1^{-1} g_2^{-1} R_{12}(\lambda) g_1 g_2. \tag{4.5}$$

Use of the (assumed to exist) operator $\log g$ allows us to explicitly solve (4.5) as

$$R_{12}(\lambda) = \exp[-\sigma(\log g_1 + \log g_2)]\big(R_{12}^0\big) \exp[\sigma(\log g_1 + \log g_2)] \tag{4.6}$$

where again σ denotes the sum over all dynamical variables $\sigma = \sum_{i=1}^n \lambda_i$ and R^0 does not depend on any variable λ_i (except as usual, in the dynamical Yang–Baxter equation, as an integer-period function). Note that in the example of [1] where $g = \exp\big[\frac{\mathrm{d}}{\mathrm{d}u}\big]$ (u spectral parameter), $R(\lambda)$ is again an exact adjoint action, so is (4.6), hence R^0 is a c-number matrix.

Consider now (2.4a). From (4.6) and (4.2) one gets

$$R_{12}^0 R_{13}^{0gg} R_{23}^0 = R_{23}^{0gg} R_{13}^0 R_{12}^{0gg} \tag{4.7}$$

hence R^0 is any non-dynamical solution of the shifted Yang–Baxter equation. It is in general not possible to go beyond this statement. However, particular solutions can easily be characterized. Any solution of the ordinary Yang–Baxter equation, commuting with $g \otimes g$ solves (4.7). In the case described in [1], for instance $g = \exp[\frac{\mathrm{d}}{\mathrm{d}u}]$, any non-dynamical matrix with a difference dependence $R_{12}(u_1 - u_2)$ solves (4.7).

To summarize, we now have the parametrized A, B, C as

$$A_{12} = b_1^{-1}\left(g_2 b_2 g_2^{-1}\right)^{-1}\left\{Ad.\left(\exp[-\sigma(\log g_1 + \log g_2)]\right) R^0_{12}\right\} g_1 b_1 g_1^{-1} \tag{4.8}$$

$$B_{12} = C_{21} = b_2^{-1} g_2 b_2(\lambda + h_1) g_2^{-1} \tag{4.9}$$

where R^0 solves (4.7). The existence of the g shift in the Yang–Baxter equation (2.4b) coupled to the dynamical 'shift' symbolized by (h_3) induces in the example in [1] a coupling between the dependence in the dynamical parameters and the spectral parameter. Indeed (4.8) will read in this case:

$$A_{12}(u_1, u_2, \lambda) = b_1(\lambda) b_2(u_2 + \gamma, \lambda) R^0_{12}(u_1 - \sigma, u_2 - \sigma) b_1(u_1 + \gamma) b_2(u_2). \tag{4.10}$$

4.2. The D-matrix and K solutions

A situation similar to section 3 arises here. Assuming first that A, B, C are parametrized as in (4.8)–(4.10) one is led to discuss whether D can be

(1) detwisted at all or not
(2) detwisted to a g-quasi non-dynamical (QND) R-matrix
(3) detwisted to a $g' \neq g$-QND R-matrix, where g' may be simply $\mathbb{1}$, or any automorphism of $\mathcal{V}$. Situations 2 and 3 may once again overlap, but this problem will not be treated here.

One again establishes immediately that

Proposition 2′. *If ABC are parametrized as in (4.8)–(4.10) by matrices b and R^0 the following two statements are equivalent:*

- *The SDYB equation (1.1) has an invertible scalar solution $k(\lambda)$.*
- *D can be detwisted according to (3.12) to a g-quasi non-dynamical R-matrix R with twist $q = gbg^{-1}k$.*

4.2.1. D detwistable. Let us first consider together cases 2 and 3 where D can be rewritten as in (3.12)

$$D_{12} = q_1^{-1}(\lambda + h_2) q_2^{-1}(\lambda) \tilde{R}_{12} q_1(\lambda) q_2(\lambda + h_1) \tag{4.11}$$

where $\tilde{R}$ is a g' quasi-non-dynamical R matrix, i.e. obeys:

$$\tilde{R}_{12}(\lambda + h_3) = g_1'^{-1} g_2'^{-1} \tilde{R}_{12} g_1' g_2'. \tag{4.12}$$

From (4.12) it now follows that $\tilde{R}$ must obey the g'-modified non-dynamical Yang–Baxter equation

$$g_1'^{-1} g_2'^{-1} R_{12} g_1' g_2' R_{13} g_2'^{-1} g_3'^{-1} R_{23} g_2' g_3' = R_{23} g_1'^{-1} g_3'^{-1} R_{13} g_1' g_3' R_{12}. \tag{4.13}$$

Eliminating now the non-trivial dynamical dependence of R implied by (4.12) we set

$$\tilde{R}_{12} = \exp -\sigma(\log g_1' + \log g_2') \bar{R}_{12} \exp \sigma(\log g_1' + \log g_2') \tag{4.14}$$

where now $\bar{R}_{12}(\lambda + h_3) = \bar{R}_{12}$, hence is independent (up to integer-periodic functions) of all dynamical variables. $\bar{R}$ also obeys the shifted non-dynamical Yang–Baxter equation (4.13) equivalent to (2.7).

Denoting now the quasi-non-dynamical R-matrices respectively by $R^{0d} \equiv \mathrm{Ad}\exp(-\sigma \log g_1 + \log g_2)R^0$ (for A) and $\tilde{R}$ (for D) and plugging the corresponding parametrizations of A, B, C, D into (1.1) one gets (denoting here by K the solution of (1.1))

$$\begin{aligned} R^{0d}_{12}(gbg^{-1}Kq^{-1})_1 q_1 (gbg^{-1}Kq^{-1})_2(h_1)q_1^{-1} \\ = (gbg^{-1}Kq^{-1})_2 q_2 (gbg^{-1}Kq^{-1})_1(h_2)q_2^{-1}\tilde{R}_{12}. \end{aligned} \tag{4.15}$$

As in (3.13) it is not easy to formulate general solutions q to (4.15). However if the conjugations $q_i X_j(h_i)q_i^{-1}$ can be trivialized, i.e. $(gbg^{-1}Kq)_i(h_j)$ is trivial on V_j, one can give explicit formulations of solutions K in terms of non-dynamical objects $\mathcal{Q}$ by eliminating all dynamical dependence between R^{0d} and $\tilde{R}$, re-expressing the equation in terms of R^0 and $\bar{R}$. Consider first the case $g' = g$.

4.2.2. D detwistable to g-QND matrix. As in the simpler case $g = \mathbb{1}$ two sets of solutions can be described:

Case 1. Non-dynamical situation

Proposition 4a. *If $\mathcal{Q}^0$ is a non-dynamical solution to the non-dynamical shifted Yang–Baxter equation:*

$$R^0_{12}\mathcal{Q}^0_1 g_2^{-1}\mathcal{Q}^0_2 g_2 = \mathcal{Q}^0_2 g_1^{-1}\mathcal{Q}^0_1 g_1 \bar{R}_{12} \tag{4.16}$$

then

$$K = gb^{-1}g^{-1}(\exp[-\sigma \log g]\mathcal{Q}^0 \exp[\sigma \log g])q \tag{4.17}$$

is also a solution of the SDR equation (1.1). If $R^0 = \bar{R}$ there exists at least one invertible $\mathcal{Q}^0 = \mathbb{1}$ and $k = gb^{-1}g^{-1}q$ provides an invertible scalar solution to (1.1) consistent with proposition 2'.

More generally one has:

Case 2: quasi non-dynamical solutions

Given an ad-factorizable automorphism a on $\mathcal{V}$ such that $[R^0, a \otimes a] = [\bar{R}^0, a \otimes a] = 0$ one also establishes

Proposition 4b. *If $\mathcal{Q}^0$ is a non-dynamical solution of the doubly shifted RTT-type equation*

$$R^0_{12}\mathcal{Q}^0_1\left(a_2^\sigma g_2 a_2^{-\sigma}\right)a_2^{-1}\mathcal{Q}^0_2 a_2\left(a_2^\sigma g_2^{-1} a_2^{-\sigma}\right) = \mathcal{Q}^0_2\left(a_1^\sigma g_1 a_1^{-\sigma}\right)a_1^{-1}\mathcal{Q}^0_1 a_1\left(a_1^\sigma g_1^{-1} a_1^{-\sigma}\right)\bar{R}_{12} \tag{4.18}$$

(where $a^\sigma = \exp[\sigma \log a]$, assuming that a has an operatorial logarithm) then

$$K = gb^{-1}g^{-1}\exp[-\sigma \log g]\exp[\sigma \log a]\mathcal{Q}^0 \exp[-\sigma \log a]\exp[\sigma \log g]q(\lambda). \tag{4.19}$$

is a solution of the SDRE (1.1).

In the particular case $\bar{R} = R^0$, (4.18) is immediately solved by $\mathcal{Q}^0 = \mathbb{1}$ hence (4.19) defines an invertible solution to SDRE. If reciprocally one can identify an invertible solution $\mathcal{Q}^{0V}$ to (4.18), then K provides an invertible solution to (1.1). As a consequence, as in the case of unshifted Yang–Baxter equations, D can be directly detwisted to

$$\tilde{R}_{12} = \left(Ad.\exp -\sigma(\log g_1 + \log g_2)R^0_{12}\right) \tag{4.20}$$

using $(Ad.\exp -(\sigma \log g)q)K$ as a twist instead of q in (4.11).

4.2.3. D detwistable to a f-QND R-matrix, f ad-factorizable. The situation becomes here rather intricate. One can however show that extensions of the two previous cases exist. Consider case 1. Equation (4.16) becomes

$$R^0_{12}\mathcal{Q}^0_1 g_2^{-1}\mathcal{Q}^0_2 f_2 = \mathcal{Q}^0_2 g_1^{-1}\mathcal{Q}^0_1 f_1 \bar{R}_{12}. \tag{4.21}$$

Solutions are then given by:

$$K(\lambda) = gb^{-1}g^{-1}\exp[-\sigma\log g]\mathcal{Q}^0\exp[\sigma\log f]q(\lambda). \tag{4.22}$$

Case 2 can be also extended to this case. The relevant equations become:

$$\begin{aligned} &R^0_{12}\mathcal{Q}^0_1\{Ad.\exp[-\sigma\log a]g\}_2^{-1}(a^{-1}\mathcal{Q}^0 a)_2\{Ad.\exp[-\sigma\log a]f\}_2 \\ &\quad = \mathcal{Q}^0_2\{Ad.\exp[-\sigma\log a]g\}_1^{-1}(a^{-1}\mathcal{Q}^0 a)_1\{Ad.\exp[-\sigma\log a]f\}_1\bar{R}_{12} \end{aligned} \tag{4.23}$$

and solutions are given by

$$K(\lambda) = gb^{-1}(\lambda)g^{-1}\exp[-\sigma\log g]\exp[-\sigma\log a]\mathcal{Q}^0\exp[\sigma\log a]\exp[\sigma\log f]q(\lambda). \tag{4.24}$$

We must make two important remarks here:

First it is important to note that in both equations (4.23) and (4.18) an explicit conjugation of the $\mathcal{V}$-automorphism g by a dynamical $\mathcal{V}$-automorphism $\exp[\sigma\log a]$ occurs. If $[a, g] = 0$ no conjugation occurs and (4.23), (4.18) are genuine non-dynamical Yang–Baxter RTT type equations for which it is consistent to search for non-dynamical solutions $\mathcal{Q}^0$. If not it may be impossible to find non-dynamical solutions $\mathcal{Q}^0$ and these cases may then be empty.

Second remark: once again if $\exp[\sigma\log g]$ or $\exp[\sigma\log f]$ are not factorizable, even though f and g are ad-factorizable, the RTT-type equations are not written as finite-size matrix algebraic equations on tensor products of the auxiliary space V. Solutions $\mathcal{Q}^0$ may then not be finite-size matrices and may not admit an expansion as formal power series of the variable u; and the object K may not be viewed as generating functional of some quantum reflection-like algebra.

4.2.4. D not detwistable. If no parametrization of D can be defined on the lines of (3.12), one can again still prove the comodule property:

Proposition 5. *If $K(\lambda)$ is a solution of (1.1) and $\mathcal{Q}$ is a non-dynamical solution of*

$$R^0_{12}\mathcal{Q}_1(g^{-1}\mathcal{Q}_2 g) = \mathcal{Q}_2(g^{-1}\mathcal{Q}_1 g)R^0_{12} \tag{4.25}$$

where R^0 and g are defined in (4.8)–(4.9), then

$$\left(g_1 b_1^{-1} g_1^{-1}\right)(\exp[-\sigma\log g]\mathcal{Q}_1\exp[\sigma\log g])g_1 b_1 g_1^{-1}K \tag{4.26}$$

is also a solution of (1.1). The dressing of an a priori given (operatorial or scalar) solution $K(\lambda)$ by suitable 'dynamical' solutions of the Yang–Baxter equation (4.25) seems to be the only available construction of new solutions in this case.

We shall now give explicit simplified formulations for the monodromy matrices obtained from the simplest comodule structures defined in [2], in the simplest parametrization context defined by proposition 4a.

4.3. Monodromy matrices when D detwistable to g-QND R

When D can be detwisted to a quasi-non-dynamical R of the same type as A the monodromy matrix built by using the comodule structure of the SDYB reflection equation, with appropriate $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, D matrices and a scalar solution $k(\lambda)$, will again simplify. Let us first consider the simplest case where D is detwisted to the same matrix $\tilde{R}$ as A, equivalent to the existence of invertible scalar solutions to (1.1). One defines the consistent parametrization:

$$A_{12} = b_1^{-1}\left(g_2 b_2 g_2^{-1}\right)^{-1} \tilde{A}_{12} b_2\left(g_1 b_1 g_1^{-1}\right) \tag{4.27}$$

$$\tilde{A}_{12} = Ad.\exp[-\sigma(\log g_1 + \log g_2)] R_{12}^0 \tag{4.28}$$

$$B_{12} = C_{12}^{\pi} = b_2^{-1} g_2 b_2(h_1) g_2^{-1} \tag{4.29}$$

$$D_{12} = k_1^{-1}(h_2) C_{12}^{-1} k_2^{-1} A_{12} k_1 B_{12} k_2(h_1). \tag{4.30}$$

Equation (4.30) just reflects the fact that since D is detwisted to $\tilde{R} = \tilde{A}$, there exists an invertible scalar solution k to (1.1) which can be used directly to rewrite D.

The monodromy matrix for a N-site chain is now defined once one stipulates a direct $\mathcal{Q}_0$ and a dual χ_0 (scalar) reflection matrix. We choose for $\mathcal{Q}_0$ the simplest parametrization described by proposition 4a when R and $\bar{R}$ are identical. To define the dual solution χ_0 we use the known identification between transposed solutions of dual SDRA, and inverse of direct solutions of SDRA, meaningful here since we know from prop. 2' that such invertible solutions exist. We set accordingly:

$$\mathcal{Q}_0 = g_0 b_0^{-1} g_0^{-1} \tilde{\mathcal{Q}}_0 g_0 b_0^{-1} g_0^{-1} k \tag{4.31}$$

$$\tilde{\mathcal{Q}} = Ad.\exp[-\sigma \log g] \mathcal{Q}_R^0, \qquad \text{where} \quad \mathcal{Q}_R^0 \text{ obeys (1.1)} \tag{4.32}$$

$$\chi_0^t = k_0^{-1} g_0 b_0^{-1} g_0^{-1} \tilde{\mathcal{Q}}_0' g_0 b_0^{-1} g_0^{-1} \tag{4.33}$$

$$\tilde{\mathcal{Q}}' = Ad.\exp[-\sigma \log g] \mathcal{Q}_L^{0-1} \qquad \text{where} \quad \mathcal{Q}_L^0 \text{ obeys (1.1).} \tag{4.34}$$

The monodromy matrix now reads [4]

$$\mathcal{T}_0 \,\mathrm{e}^{\partial_0} = \chi_0^t g_0 A_{02N} g_{2N}^{-1} g_0 C_{02N-1} g_0 A_{02N-2}(h_{2N-1}) g_{2N-2}^{-1} \cdots \tag{4.35}$$

$$\cdots \mathcal{Q}_0(h_1 + h_2 + \cdots h_{2N-1}) D_{01}(h_1 + h_2 + \cdots h_{2N-1}) \cdots g_{2N} B_{02N} \,\mathrm{e}^{\partial_0} \tag{4.36}$$

Remark. Contrary to the scalar k matrix, the matrix $\mathcal{T}_0 \,\mathrm{e}^{\partial_0}$ exhibits a non-adjoint action of g_0 (but an adjoint action of all non-zero indexed operators g_i). This may lead to a fundamental problem.

In the non-affine case, when $\dim \mathcal{V} \equiv V < \infty$ the transfer matrix is defined as a trace over V hence no difficulty arises. If however $\mathcal{V}$ is an evaluation module $V \otimes \mathbb{C}[[u]]$, one is actually interested in partial traces over V to define spectral-parameter dependent transfer matrices $Tr_V(\mathcal{T}_0 \,\mathrm{e}^{\partial_0})$. In this case if g acts non-trivially on $\mathbb{C}[[u]]$, more specifically if g is not factorizable (as in [1] where $g = \exp[\frac{\mathrm{d}}{\mathrm{d}u}]$) the proof of commutation of such partial traces using the $A\mathcal{T}B\mathcal{T}$ relations is not valid, as can be seen on our example since the $\mathcal{T}$ matrices will then contain explicit operators $\exp\left[\frac{\mathrm{d}}{\mathrm{d}u}\right]$ acting on matrix elements of A, B, C, D! As a matter of fact even the partial traces over such monodromy matrices do not exist since the matrices themselves do not assume the factorized form of $\dim((V)^{\otimes N} \otimes V)$-size matrices depending on $N + 1$ spectral parameters.

A solution to this issue is the following: one has to assume that D, B, C exhibit the same zero-weight properties under the adjoint action of g as they already did, as a fundamental assumption of our semi-dynamical structure, under the adjoint action of $\mathfrak{h}$. In addition one will assume that g and $\mathfrak{h}$ commute:

$$[D, g \otimes g] = [B_{12}, g \otimes \mathbb{1}] = [C_{12}, \mathbb{1} \otimes g] = 0 \qquad [h, g] = 0. \tag{4.37}$$

This situation is indeed realized in [1] since (4.37) here immediately follows from the particular dependence of D, B and C on the spectral parameter: $g = \exp\left[\frac{\mathrm{d}}{\mathrm{d}u}\right]$ and $D_{12} = D_{12}(u_1 - u_2)$, $B_{12} = B_{12}(u_2)$ and $C_{12} = C_{12}(u_1)$. Adjoint action of g is simply shift of the corresponding spectral parameter.

Once (4.37) is imposed it is easy to prove:

Proposition 7. *If K is a solution to (1.1), Kg^n is a solution to (1.1) for any integer $n \in \mathbb{Z}$.*

The monodromy matrix (4.36) can then be modified to take the form of an exact adjoint action (hence factorizable) :

$$\mathcal{T}_0\, \mathrm{e}^{\partial_0} \to \mathcal{T}_0\, \mathrm{e}^{\partial_0} g_0^{-2N}. \tag{4.38}$$

Since we have restricted g to be ad-factorizable, the partial trace of the monodromy matrix is now once again correctly defined; its expansion in formal power series of u is also defined and generates commuting Hamiltonians.

Let us make here a technical remark: locality conditions on these Hamiltonians may then be imposed (see [4]) and lead to specific choices of the values of the quantum-space spectral parameters. As a particular example let us point out that in the case treated in [1], the shifts in (4.36) are distributed according to:

$$\begin{aligned}&\cdots A_{02n}(\lambda_0 + (1 + 2N - 2n), \lambda_{2n}) C_{02n-1}(\lambda_0 + (2 + 2N - 2n), \lambda_{2n-1})\\ &\cdots D_{02n-1}(\lambda_0 + (2N), \lambda_{2n-1}) B_{02n}(\lambda_{2n})\end{aligned} \tag{4.39}$$

and the locality conditions on the Hamiltonians have a consistent implementation as $\lambda_{2n} = \lambda_0 + (2N - 2n + 1)$, $\lambda_{2n-1} = \lambda_0 + 2N$.

If these assumptions are realized, plugging now (4.27), (4.29), (4.31), (4.33) into (4.36) yields

$$\mathcal{T}_0\, \mathrm{e}^{\partial_0} = \mathcal{O}_N^{-1} \tilde{\mathcal{T}}_0\, \mathrm{e}^{\partial_0} \mathcal{O}_N \tag{4.40}$$

where

$$\tilde{\mathcal{T}}_0 = k_0^{-1} g_0 b_0^{-1} g_0^{-1} \tilde{\mathcal{Q}}_0' g_0 \tilde{A}_{02N} \cdots g_0 A_{02} \tilde{\mathcal{Q}}_0 g_0 \tilde{A}_{01} \cdots g_0 \tilde{A}_{02N-1} \big(g_0 b_0 g_0^{-1}\big) k_0 g_0^{-2N} \tag{4.41}$$

$$\begin{aligned}\mathcal{O}_N = \Pi_{m=0}^{N-1} &\big(g_{2N-2m} b_{2N-2m}\big(h_<^{\mathrm{odd}}\big) g_{2N-2m}^{-1}\big)\big(g_{2N-2m-1} b_{2N-2m-1}\big(h_<^{\mathrm{odd}}\big)\\ &\times g_{2N-2m-1}^{-1} k_{2N-2m-1}\big(h_<^{\mathrm{odd}}\big)\big)\end{aligned} \tag{4.42}$$

Reformulating the quasi-non-dynamical $\tilde{A}$, $\tilde{\mathcal{Q}}_0$ and $\tilde{\mathcal{Q}}_0'$ in (4.41) following (4.27)–(4.34) one finally gets

$$\begin{aligned}\tilde{\mathcal{T}}_0 = k_0^{-1} g_0 b_0^{-1} g_0^{-1} &\exp[-\sigma \log g_0] \mathcal{Q}_L^{0-1} Ad \exp[-\sigma(\log g_1 + \cdots g_{2N})]\\ &\big\{g_0 R_{02N}^0 \cdots g_0 R_{02}^0 \mathcal{Q}_R^0 g_0 R_{01}^0 \cdots g_0 R_{02N-1}^0\big\} \exp[\sigma \log g_0] g_0 b_0 g_0^{-1} k_0 g_0^{-2N}.\end{aligned} \tag{4.43}$$

Comment: $\tilde{\mathcal{T}}_0$ is therefore decomposed as a non-dynamical chain monodromy matrix with direct/dual ‘reflection’ matrix dressed dynamically by the shift-dynamical coupling $Ad.\exp[-\sigma \log g_0]$; more fundamentally dressed by the adjoint action of the Drinfeld twist $g_0 b_0 g_0^{-1} k_0$, which turns D into $\tilde{R}$), yielding a generating functional for the commuting Hamiltonians by the dynamical trace formula $Tr_0(\tilde{\mathcal{T}}_0 \exp[\partial_0])$.

4.4. Monodromy matrices when D detwistable to $\bar{R}$ not equivalent to R

Here one must substitute into (4.30) the general twisting relation (4.11). Using now as a direct reflection matrix a solution of the form (4.17) and a (non-parametrized) dual reflection matrix χ_0^t one gets for the monodromy matrix a formula analogous to (4.43) with the following modifications:

(1) The blocks $(gbg^{-1}k)$ in $\mathcal{O}$ and the term $g_0b_0g_0^{-1}k_0$ on the rhs of (4.43) must be substituted by the twist matrix q from D to $\bar{R}$.
(2) Odd-labelled R_{02k+1} are substituted by $\bar{R}_{02k+1}$ defined in (4.14).
(3) Since no invertible solution to (4.16) exists here, we cannot identify a dual solution with any 'inverse' of a direct solution. Parametrization (4.33), (4.34) is however still valid provided that $k_0^{-1}g_0b_0^{-1}g_0^{-1}$ be replaced by q_0^{-1} and $\mathcal{Q}_L^{0-1}$ by an explicitly computed solution of the transposed dual equation to (4.16). This transposed dual equation is trivially obtained by taking the formal inverse of (4.16). The term $k_0^{-1}g_0b_0^{-1}g_0^{-1}$ on the l.h.s. of (4.43) is consequently to be substituted by q_0^{-1}.

It is not clear whether such transfer matrices are useful to build physically interesting spin-chain-type models. Their explicit formulation however may be interesting in itself to understand the algebraic structures underlying (1.1) in the non-trivial case where A, B, C and respectively D yield distinct R matrices.

4.5. Remarks on the structure of monodromy matrices

As commented upon in the previous sections, the monodromy matrices take a very characteristic form once the parametrization of A, B, C, D, k and k^{dual} is taken into account. One identifies first non-dynamical chain transfer matrices with direct and dual scalar Lax matrix q_R and q_L, which would yield by the standard construction closed spin-chain Hamiltonians. They are then dressed non-trivially by the adjoint action of the Drinfeld twist q, which characterizes the D matrix, and the subsequent generating functional for commuting Hamiltonians is

$$t(\lambda) = Tr_0\{q_0^{-1}(\lambda)T_0q_0(\lambda)\,\mathrm{e}^{\partial_0}\} \tag{4.44}$$

The key remark here is that mutual commutation of such objects with different spectral parameters u_0, u_0', or of 'quantum trace-like' objects obtained from fusion procedures on the auxiliary space ((0)-index) as was derived in [3], is guaranteed by the necessary conditions on the twist q_0, i.e. that the D-matrix obtained as dynamical twist of the non-dynamical or quasi-non-dynamical R-matrix in $\mathcal{T}_0$ as:

$$D_{12} = q_2^{-1}(h_1)q_1R_{12}q_2q_1(h_2) \tag{4.45}$$

have zero weight. Otherwise the $(\exp[\partial_0])$ term prevents commutation of the generating functions $([t(u), t(u')] = 0)$. Remarkably though, zero weight condition on D is also a sufficient condition (proposition 1) to guarantee that D obeys the dynamical Yang–Baxter equation.

This leads us to conclude that the semi-dynamical 'reflection' equation is not really a 'reflection' equation, in the usual sense of the term, since in any case B and C have non-canonical, loosely speaking 'semi-diagonal', zero-weight conditions. It seems that one underlying fundamental structure is the dynamical Yang–Baxter algebra (dynamical quantum group) associated with the matrix D; the decomposition (4.45) is then used to build dynamical monodromy matrices (4.44) although bypassing the zero-quantum weight requirement [7], which occurs when using directly Lax matrices of the dynamical quantum group to build monodromy matrices. This requirement is eliminated by the trick of building a reflection-type quadratic exchange algebra with no dynamical shifts in the coefficient matrices. The SDRE is therefore an intermediate construction between the non-dynamical quantum group (R-matrix) and the dynamical quantum group (D-matrix). Its main practical interest is that it naturally yields a dynamical un-constrained (no-zero weight) monodromy matrix (4.44). Let us once again remark that in any case (4.44) does not admit an obvious interpretation as a trace in the quantum group. The groupoid formulation, advocated in [11], and quite naturally adapted to dynamical R matrices may provide a natural framework for (4.44).

5. Conclusions and perspectives

5.1. The spin-chain Hamiltonians

Traces over the auxiliary space (labelled by 0) of the monodromy matrices such as described in sections 3.3 and 4.3 provide a systematic way of constructing quantum integrable Hamiltonians [4].

It is first essential to remark that in this construction the quantum-space operators $\mathcal{O}_N$ are not relevant to keep since they simply conjugate the quantum monodromy matrix, and the Hamiltonians deduced from it. One must therefore realize the computation of quantum integrable Hamiltonians from the non-conjugated monodromy matrix, thereby eliminating all cumbersome quantum-space shifts.

These now factored-out monodromy matrices exhibit a very interesting combination of features. The untwisted part $R \ldots \mathcal{Q} \ldots R$, as already mentioned, has the canonical form of a generating functional (once taking the trace over the auxiliary space) for closed spin-chain Hamiltonians. The zero-site twisted monodromy matrix, built from a single scalar reflection matrix k and a trivial dual solution $\mathbb{1}$, yields precisely scalar RS Hamiltonians when choosing $ABCD$ structure matrices from [1]. The question of how such features interplay in the new generated Hamiltonians to yield possible 'spin Ruijsenaar Schneider models' is therefore quite challenging. More precisely the procedure should now run as follows.

In the non-affine case (no spectral parameter) the trace over the full auxiliary space V is expected to yield Hamiltonians of N-body systems with interactions a la Ruijsenaar–Schneider. A family of commuting higher-degree Hamiltonians can then be obtained by a now well-established quantum trace procedure, see e.g. [3, 7].

Consider now the more interesting case of affine SDRA where $\mathcal{V} = V \otimes \mathbb{C}[z]$. It was shown in [4] how quantum integrable 'spin RS' Hamiltonians could be obtained by the canonical procedure of taking the logarithmic derivative of the partial trace over V of the dynamical monodromy matrix w.r.t. the spectral parameter z_0 associated with the auxiliary space, at $z_0 = 0$.

The generalized 'spin–spin' interactions then take a local form (nearest neighbour or next-to-nearest neighbour interaction) for a suitable consistent choice of the values of the 'quantum' spectral parameters $z_{1\ldots 2N}$, provided that the structure matrices A and D obey the so-called regularity conditions $A(z_1 = 0, z_2 = 0) = D(z_1 = 0, z_2 = 0) = P_{12}$, where P_{12} is the permutation operator on $V \otimes V$. This suitable choice was commented on in section 4, see (4.39).

The technical problem which arose in the previous approach [4] when directly computing the form of these Hamiltonians lied in the complexity of the formulae once written in terms of non-parametrized matrices A, B, C, D. Reformulating the structure matrices as we have done, the new factorized monodromy matrices are essentially formulated in terms of one single non-dynamical R-matrix R and one consistently associated twist matrix q yielding a dynamical D matrix through (2.5). The affine case is however the situation where [16] in principle does not extend, and q-matrices must be explicitly constructed 'by hand' from given D-matrices. It is *a priori* known that they exist for the specific RS $ABCD$ matrices since in this case all d_{ij} are non-zero, all b_i are invertible, and $k(\lambda) = \mathbb{1}$ is known to be a solution. Proposition 2' then applies; the final step to get explicit Hamiltonians is now to compute matrices q and R from known non-constant D matrices and derive explicit 'spin-chain RS'-type Hamiltonians from the factorized forms (4.43).

The specific form of the interaction will also depend on the choice of the scalar solution k. Classification of solutions k for the non-affine rational case of [1] is now fully known [5], and classification in the affine rational case is currently in progress.

We finally want to indicate that relaxing some technical hypothesis, such as non-factorizability, may yield interesting generalizations of the RS type Hamiltonians[2].

5.2. *Connection between SDRA and quantum groups*

In the most regular situations, when $B = C^{\pi}$ is invertible, $d_{ij} \neq 0$ for all i, j and SDRE (1.1) has at least one invertible scalar solution $k(\lambda)$, the simplest parametrization of solutions K proposed as subset 3.2.2a1 allows us to prove that any representation of the ordinary quantum group ($RTT = TTR$) generates a representation of the SDRA. The inference is only one-sided since one cannot preclude the possibility of dynamical solutions to the reduced equation (3.13). In addition the monodromy matrices generated by the simplest representations of the comodule structure by $ABCD$ matrices are expressed in terms of the standard monodromy matrix generated by the ordinary quantum group R-matrix, which yields closed spin-chain Hamiltonians. In this respect one can say that the twisting procedure is compatible with the comodule structure, which is of course to be expected if they represent universal algebraic structures (Drinfel'd twist and coproduct) which would underlie the SDRA.

The form of the dynamical trace $\mathrm{Tr}\{\mathcal{T}\,\mathrm{e}^{\partial_0}\}$ however remains a specific feature of the SDRE, and—as already commented—does not naturally yield an element of the SDRA itself, but may rather be understood in terms of a more complex algebraic structure, possibly a quantum groupoid [11, 20].

These conclusions can be extended to the 'g-modified' extension of the SD Yang–Baxter equations. The 'g-modified quantum group' structure $R_{12}T_1T_2^g = T_2T_1^gR_{12}$ however is a less standard one and certainly deserves more exploration, in particular since it is the relevant one when considering the elliptic Ruijsenaars-Schneider example developed in [1].

In the non-regular situation, when no invertible $k(\lambda)$ is available, the representations of intertwining relations $R_{12}T_1T_2^g = T_2T_1^gR_{12}$ are now relevant to build representations of the corresponding SDRA, and monodromy matrices. In fact, as mentioned before, the SD'R'E (1.1) is not so much defining a reflection algebra as providing an intertwining formulation between a conjugated R-matrix A and a dynamical twisted D matrix with same or different underlying non-dynamical or quasi-non-dynamical R-matrices, themselves associated with quantum group-like algebraic structures. A better understanding of this structure may require a (partial) lifting of the sufficient conditions, e.g. the (quasi) non-dynamicity condition on gKg^{-1}. In addition, lifting the conditions of B-invertibility or $d_{ij} \neq 0$ may provide interesting non-trivial new examples.

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[2] This intriguing possibility was suggested to us by one referee.

Appendix A. Semi-dynamical quantum reflection algebra

Quantum reflection algebras were first formulated in [21, 22] as consistency conditions between factorizable 2-body S-matrices of quantum integrable systems, and 1-body reflections K-matrix, guaranteeing the quantum integrability of the system with boundaries. They take the general form

$$A_{12}K_1B_{12}K_2 = K_2C_{12}K_1D_{12}. \tag{A.1}$$

Equations (A.1) are now interpreted as quadratic constraint equation for generators of the quantum algebra $\mathcal{G}$ encapsulated in the matrix K. It is represented as an equation in $\mathrm{End}(\mathcal{V}) \otimes \mathrm{End}(\mathcal{V})$ with elements in $U(\mathcal{G})$ where $\mathcal{V}$ is a given vector space known as the auxiliary space. $ABCD$ are matrices in $\mathrm{End}(\mathcal{V}) \otimes \mathrm{End}(\mathcal{V})$. $\mathcal{V}$ may be—in the most usual case—a finite vector space V or a loop vector space $V \otimes \mathbb{C}[[z]]$ (the abstract formal variable z being the so-called spectral parameter). However one may retain the possibility that $\mathcal{V}$ be a more general vector space (functional space), even though it will not be considered in the present work. K now belongs to $\mathrm{End}(\mathcal{V}) \otimes \mathcal{G}$. A generalized quantum reflection algebra may be defined when assuming that $ABCD$ and K depend on a further set of complex variables, collectively denoted by $\lambda = \{\lambda_i, i = 1, \ldots, n\}$, interpreted as coordinates on the dual of a characteristic (usually Abelian) complex Lie algebra of finite dimension, and parametrizing a deformation of (A.1). This is in fact an extension to (A.1) of the so-called dynamical deformation of YB equation defined in [14, 15, 23, 24] where the YB equation is originally introduced as

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{A.2}$$

$$R_{12}(\lambda + h_3)R_{13}R_{23}(\lambda + h_1) = R_{23}R_{13}(\lambda + h_2)R_{12}. \tag{A.3}$$

Here since λ are coordinates on the dual of $\mathfrak{h}$, it is understood that the auxiliary space $\mathcal{V}$ is an irreducible diagonalizable module of $\mathfrak{h}$, justifying the notation '$\lambda + h_i$'. 'Irreducibility' is an extra requirement, implying that zero-weight matrices under adjoint action of $\mathfrak{h}$ necessarily admit an expansion of the finite generators of $\mathfrak{h}$, which will be very useful in all our discussions.

In fact two dynamical extensions of the RA (A.1) have now been identified. The semi-dynamical RA, which interests us here, reads:

$$A_{12}K_1B_{12}K_2(\lambda + h_1) = K_2C_{12}K_1(\lambda + h_2)D_{12}. \tag{A.4}$$

The fully dynamical RA or 'boundary dynamical RA' [25, 26] reads

$$A_{12}K_1(\lambda + h_2)B_{12}K_2(\lambda + h_1) = K_2(\lambda + h_1)C_{12}K_1(\lambda + h_2)D_{12}. \tag{A.5}$$

Appendix B. The irreducibility criterion in the affine case

We have chosen two specific cases for the auxiliary space $\mathcal{V}$, either as a finite-dimensional vector space $\mathcal{V} = V$, or as a loop space $\mathcal{V} = V \otimes \mathbb{C}[z]$. The finite-dimensional vector space V is in addition assumed to be a diagonalizable irreducible module for the dynamical Lie algebra $\mathfrak{h}$, allowing us in this way to consistently expand any zero weight matrices on any basis of $\mathfrak{h}$, e.g. the basis of normalized diagonal n-matrices $\{e_{ii}\}$ (here $(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$) when $\mathfrak{h} = \mathrm{Cartan}(gl(n))$.

It would seem that therefore, when $\mathcal{V} = V \otimes \mathbb{C}[z]$, the full auxiliary space $\mathcal{V}$ is no more an irreducible module of $\mathfrak{h}$. However if $\mathfrak{h}$ is completed by the derivation generator d—as it should be when considering affine Lie algebras—represented as $d = \frac{\mathrm{d}}{\mathrm{d}u}$, $\mathcal{V}$ is again irreducible. One would expect in this case occurrence of an $(n+1)$th coordinate λ_d with a dynamical shift in (1.1). However, in the known case of dynamical elliptic quantum groups [14] the dynamical shift on the coordinate associated with d is interpreted as the central charge c in a centrally extended dynamical quantum algebra, hence it is set to 0 in an evaluation representation. Since the shifts in the definition of the dynamical reflection algebra (1.1) occur precisely on the auxiliary spaces, the absence of an explicit $(n+1)$th shift in (1.1) does not contradict the existence of a (here non-relevant) extra variable (such as the elliptic module p in an elliptic DRA) and thus the interpretation of (1.1) as dynamical quantum reflection algebra, with dynamical Lie algebra $\mathfrak{h} \cup \{d\}$, for which $\mathcal{V}$ is again an irreducible module. Note that the choice of $\hat{\mathfrak{h}} = \mathfrak{h} \cup \{d\}$ as underlying Abelian Lie algebra defining the dynamical deformation now implies—for consistency of the construction—to implement full $\hat{\mathfrak{h}}$ zero-weight conditions on B, C, D, i.e. including the adjoint action of d. In this case $g = d$ becomes a suitable automorphism, under the conditions in (4.37) to build monodromy matrices in the g-deformed YB frame. This is precisely the situation realized in [1].

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